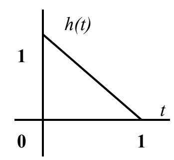
Convolution

Lecture #6

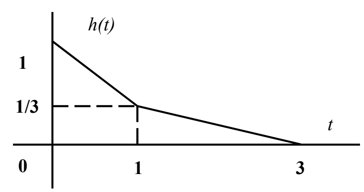
2CT.3 – 8

Homework

- Convolution Verify your all your results of these convolution problems using Matlab and its conv function.
  - Problem (1)
    - Assume that a system response is given by the following:



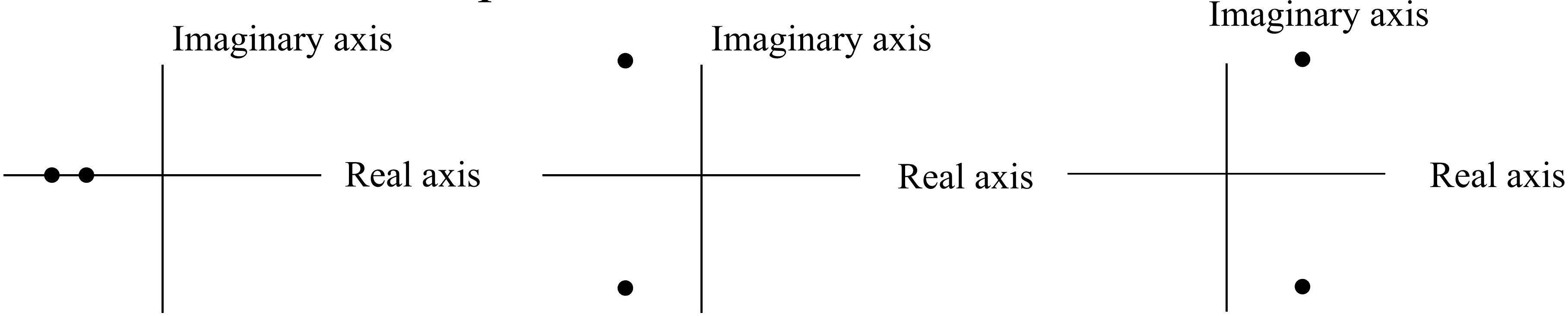
    - Sketch the response to a) *u(t),* b) *u(t)-u(t-a)* for a=0.5, a=1, and a=5, and c) evaluate $e^{-t}\,u(t)$ at *t=1* and *t=2*
  - Problem (2)
    - Assume that a system response is given by the following:



    - Evaluate the response to $te^{-t}\,u(t)$ at *t=1* and *t=3*

Homework

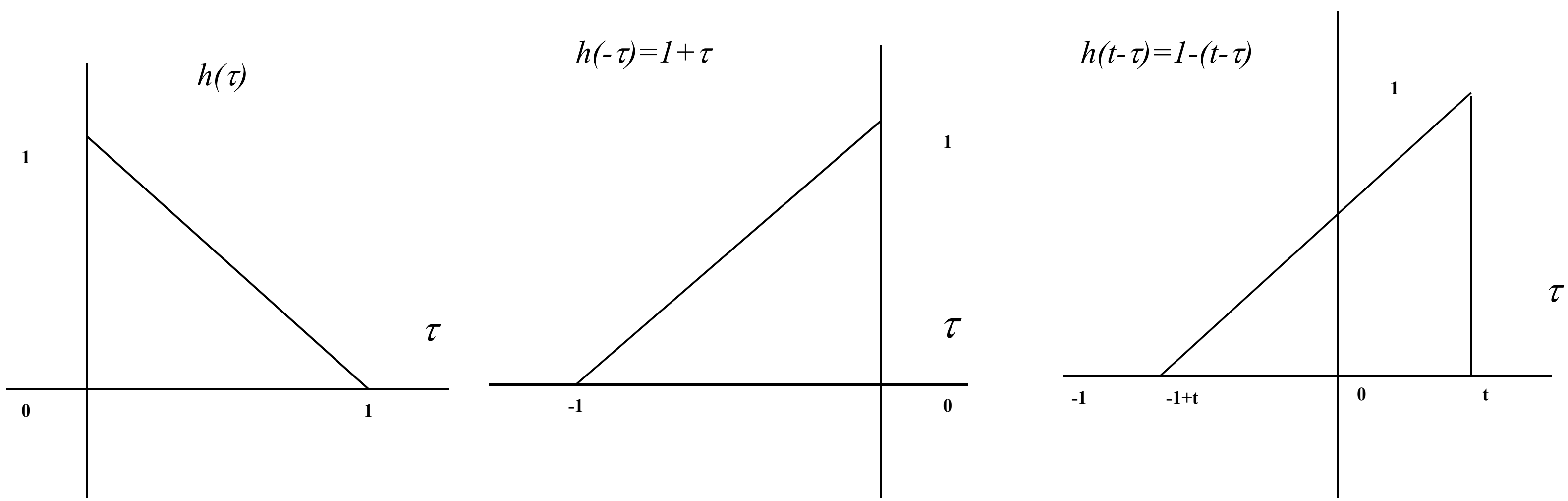
- Stability
  - Determine the stability of the following systems with poles in the complex plane, describe the form of the transient response:



- 2CT.3.1, 2CT.3.2

Homework Answers #1

- Convolution
  - Problem (1)
    - Assume that a system response is given by the following:



$$h(\tau) = (1-\tau) \text{ for } 0 < \tau < 1$$

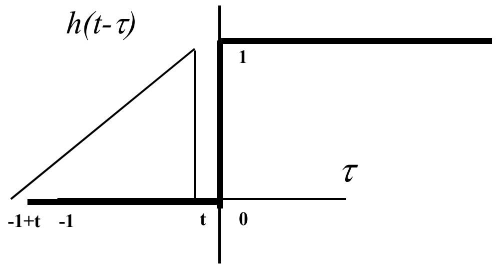
$$h(-\tau) = (1+\tau) \text{ for } -1 < \tau < 0$$

$$h(t-\tau) = [1-(t-\tau)] \text{ for } -(1-t) < \tau < t$$

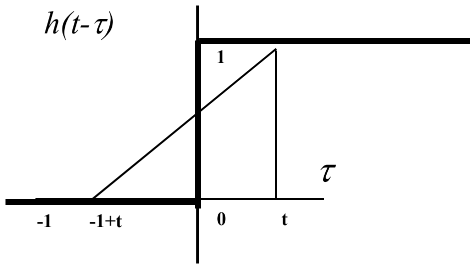
Homework Answers #2

- Convolution
  - Problem (1)
    - Sketch the response to a) *u(t)*

$$a)\ C = \int [1-(t-\tau)]u(t-\tau)u(\tau)d\tau$$



t<0, C=0



$$0 < t < 1$$

$$C = \int_0^t 1*[1-(t-\tau)]d\tau = \left.\frac{[1-(t-\tau)]^2}{2}\right|_0^t$$

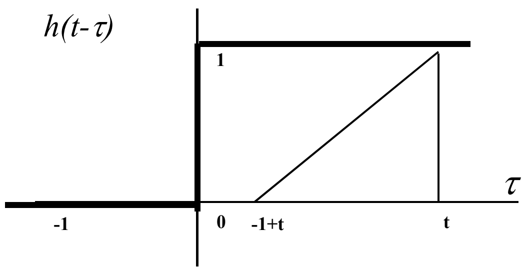
$$= \frac{1}{2} - \frac{[1-t]^2}{2} = \frac{1}{2}[1-(1-t)^2]$$

$$= \frac{1}{2}[1-1+2t-t^2] = t - \frac{t^2}{2}$$

Homework Answers #3

- Convolution
  - Problem (1)
    - Sketch the response to a) *u(t)*

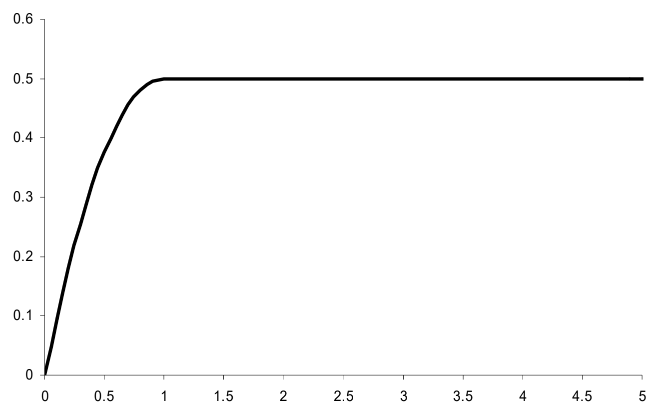
$$a)\ C = \int [1-(t-\tau)]u(t-\tau)u(\tau)d\tau$$



$$t>1,\ C = \int_{-1+t}^{t} 1\times[1-(t-\tau)]d\tau$$

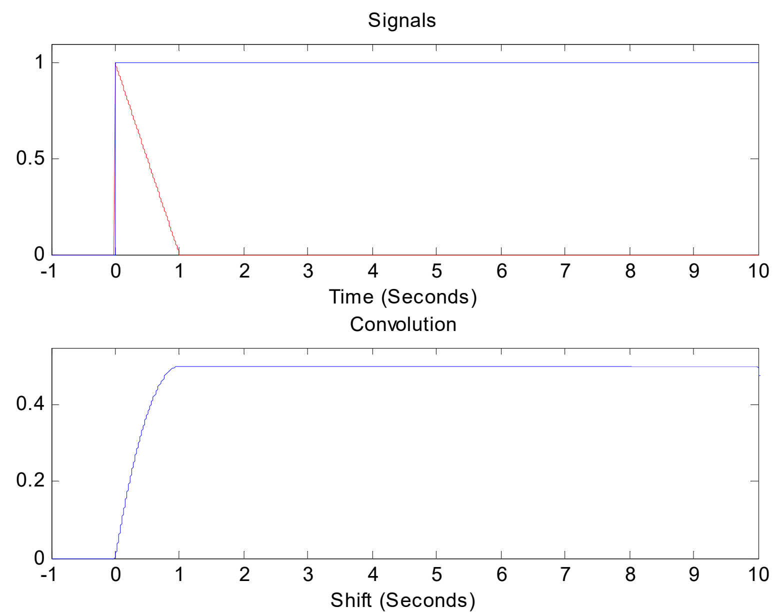
$$= \left.\frac{[1-(t-\tau)]^2}{2}\right|_{-1+t}^{t}$$

$$= \frac{1}{2} - \frac{0}{2} = \frac{1}{2}$$



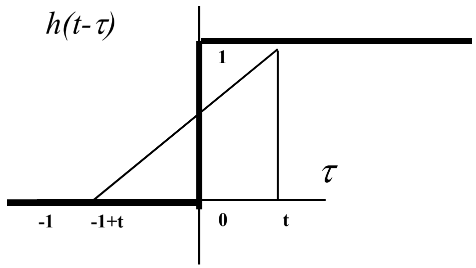
Matlab Code

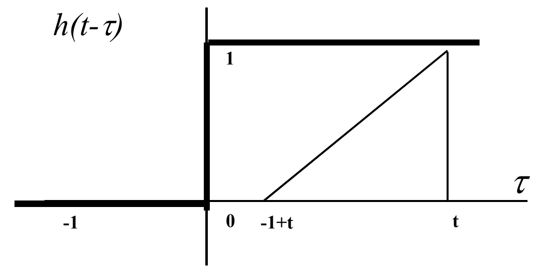
```
clear all;
endpulse=20;
ts=.001;
endpoint=10;
n=-endpoint:ts:endpoint;
nn=-endpoint*2:ts:endpoint*2;
pulser=(n>=0)&(n<=endpulse);
pulse=1*pulser;
tripulse=(n>=0)&(n<=1);
tri=(1-n).*tripulse;
subplot(2,1,1)
plot(n,tri,'r',n,pulse,'b');
title('Signals');
xlabel('Time (Seconds)');
axis([-1 10 min([min(tri) min(pulse)]) 1.1*max([max(tri) max(pulse)])]);
c=conv(pulse,tri)*ts;
subplot(2,1,2)
plot(nn,c);
title('Convolution');
xlabel('Shift (Seconds)');
axis([-1 10 min(c) 1.1*max(c)]);
```



Homework Answers #2

- Convolution
  - Problem (1) Alternative integration
    - Sketch the response to a) *u(t)*





$0 < t < 1$

$$C = \int_0^t 1*[1-(t-\tau)]d\tau = [\tau + \frac{\tau^2}{2} - \tau t]\Big|_0^t$$

$$= t + \frac{t^2}{2} - t^2 = t - \frac{t^2}{2}$$

$$t > 1,\ C = \int_{-1+t}^{t} 1\times[1-(t-\tau)]d\tau$$

$$= \tau + \frac{\tau^2}{2} - \tau t\Big|_{-1+t}^{t} = t + \frac{t^2}{2} - t^2 - [(t-1) + \frac{(t-1)^2}{2} - (t-1)t]$$

$$= t + \frac{t^2}{2} - t^2 + [(-t+1) - \frac{(t-1)^2}{2} + (t-1)t]$$

$$= t + \frac{t^2}{2} - t^2 + [-t+1 - \frac{t^2}{2} + \frac{2t}{2} - \frac{1}{2} + t^2 - t]$$

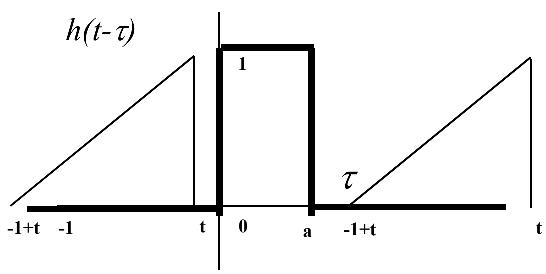
$$= t - \frac{t^2}{2} + [\frac{t^2}{2} + \frac{1}{2} - t] = \frac{1}{2}$$

Homework Answers #4

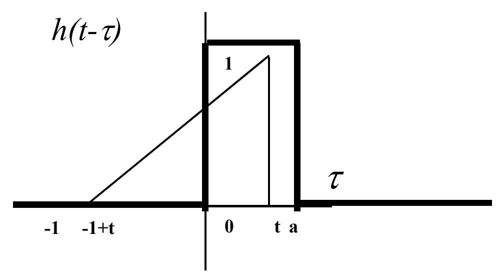
- Convolution
  - Problem (1)
    - Sketch the response to b) *u(t)-u(t-a)* for a=0.5

$$b)\ \mathrm{C} = \int [1-(t-\tau)]u(t-\tau)u(\tau)d\tau + \int [1-(t-\tau)]u(t-\tau)u(\tau-a)d\tau$$

for $a \leq 1$



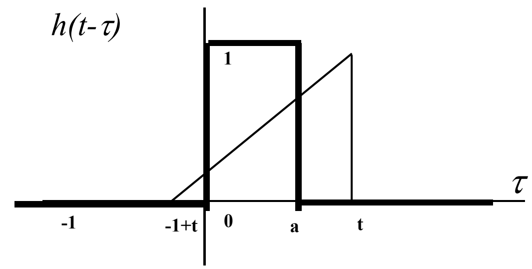
$t < 0$ & $t > a+1$,
C=0



$0 < t < a$

$$\mathrm{C} = \int_0^t 1*[1\text{-}(t-\tau)]d\tau = \left.\frac{[1\text{-}(t-\tau)]^2}{2}\right|_0^t$$

$$= \frac{1}{2} - \frac{[1\text{-}t]^2}{2} = \frac{1}{2}[1-(1-t)^2]$$



$a < t < 1$

$$\mathrm{C} = \int_0^a 1*[1\text{-}(t-\tau)]d\tau$$

$$= \left.\frac{[1\text{-}(t-\tau)]^2}{2}\right|_0^a$$

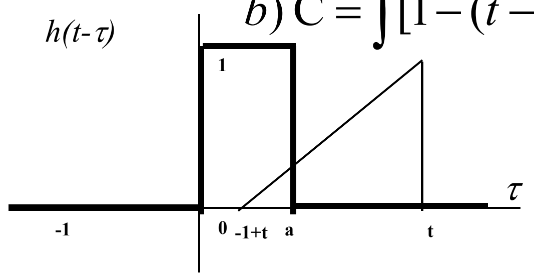
$$= \frac{[1\text{-}(t-a)]^2}{2} - \frac{[1\text{-}t]^2}{2}$$

$$= \frac{2a-2at+a^2}{2}$$

Homework Answers #5

- Convolution
  - Problem (1)
    - Sketch the response to b) *u(t)-u(t-a)* for a=0.5

$$b)\ C = \int [1-(t-\tau)]u(t-\tau)u(\tau)d\tau + \int [1-(t-\tau)]u(t-\tau)u(\tau-a)d\tau$$



for $a \leq 1$

$$t < a+1$$

$$C = \int_{-1+t}^{a} 1*[1-(t-\tau)]d\tau$$

$$= \frac{[1-(t-\tau)]^2}{2}\Big|_{-1+t}^{a}$$

$$= \frac{[1-(t-a)]^2}{2} - 0$$

$$= \frac{[1-(t-a)]^2}{2}$$

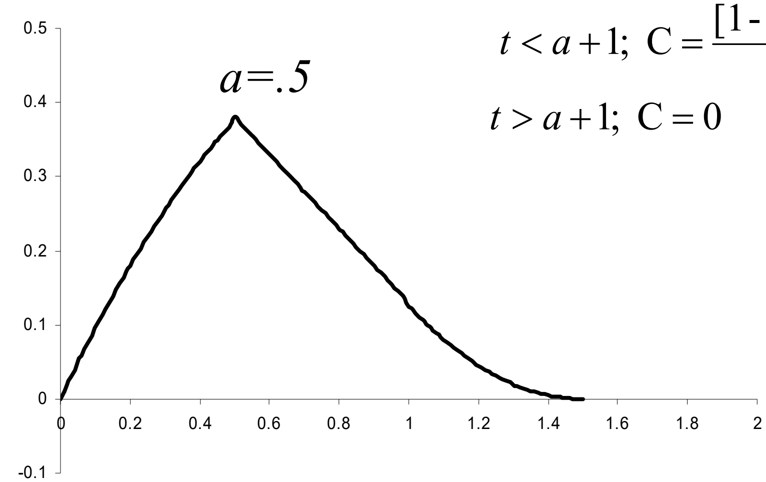
$t<0;\ C=0$

$0<t<a;\ C=\frac{1}{2}[1-(1-t)^2]$

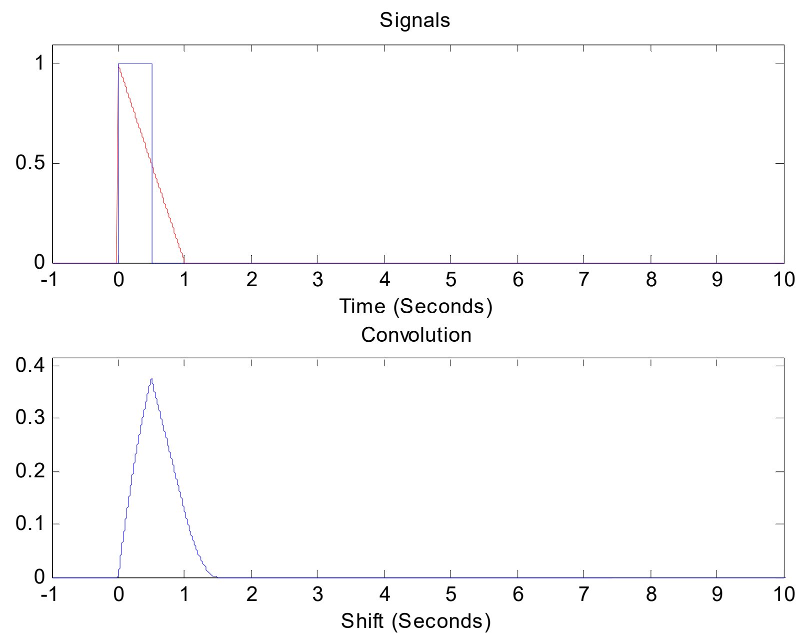
$a<t<1;\ C=\frac{2a-2at+a^2}{2}$

$t<a+1;\ C=\frac{[1-(t-a)]^2}{2}$

$t>a+1;\ C=0$



Matlab Code




```
clear all;
endpulse=.5;
ts=.001;
endpoint=10;
n=-endpoint:ts:endpoint;
nn=-endpoint*2:ts:endpoint*2;
pulser=(n>=0)&(n<=endpulse);
pulse=1*pulser;
tripulse=(n>=0)&(n<=1);
tri=(1-n).*tripulse;
subplot(2,1,1)
plot(n,tri,'r',n,pulse,'b');
title('Signals');
xlabel('Time (Seconds)');
axis([-1 10 min([min(tri) min(pulse)]) 1.1*max([max(tri) max(pulse)])]);
c=conv(pulse,tri)*ts;
subplot(2,1,2)
plot(nn,c);
title('Convolution');
xlabel('Shift (Seconds)');
axis([-1 10 min(c) 1.1*max(c)]);
```

Homework Answers #6

- Convolution
  - Problem (1) :
    - Sketch the response to b) *u(t)-u(t-a)* for a=1

$$b)\ \mathrm{C} = \int [1-(t-\tau)]u(t-\tau)u(\tau)d\tau + \int [1-(t-\tau)]u(t-\tau)u(\tau-a)d\tau$$

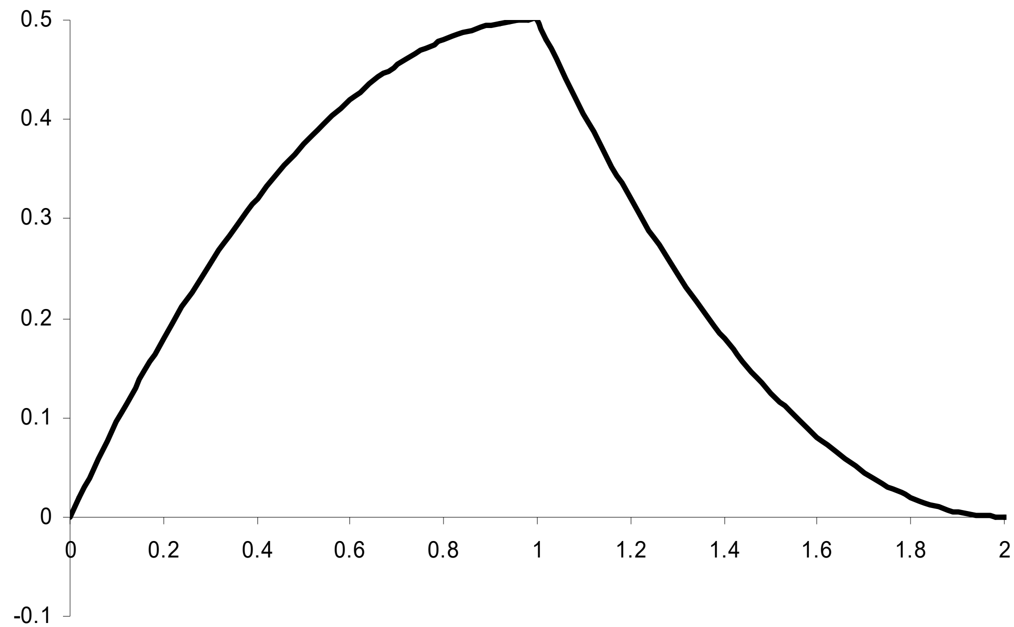
$$\text{for } a \leq 1$$

$t < 0;\ C = 0$

$0 < t < 1; \mathrm{C} = \frac{1}{2}[1-(1-t)^2]$

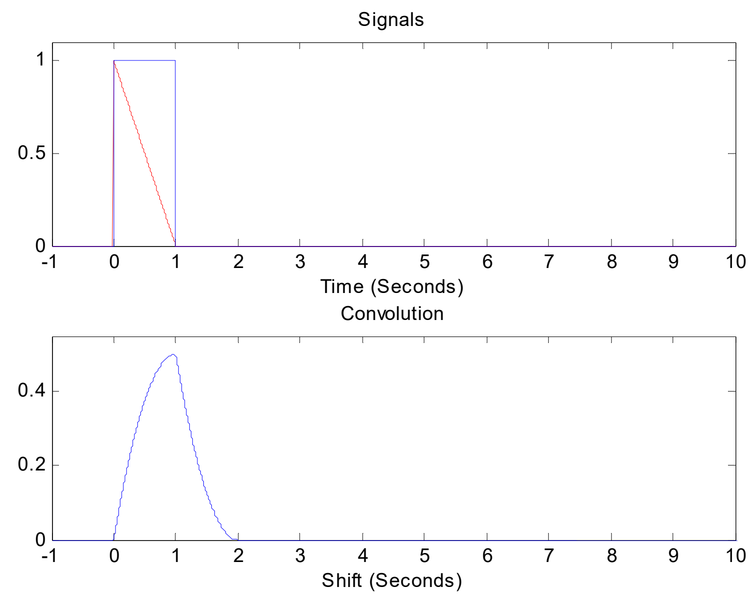
$t < 2;\ \mathrm{C} = \frac{[1\text{-}(t-1)]^2}{2}$

$t > 2;\ \mathrm{C} = 0$



Matlab Code

```
clear all;
endpulse=1;
ts=.001;
endpoint=10;
n=-endpoint:ts:endpoint;
nn=-endpoint*2:ts:endpoint*2;
pulser=(n>=0)&(n<=endpulse);
pulse=1*pulser;
tripulse=(n>=0)&(n<=1);
tri=(1-n).*tripulse;
subplot(2,1,1)
plot(n,tri,'r',n,pulse,'b');
title('Signals');
xlabel('Time (Seconds)');
axis([-1 10 min([min(tri) min(pulse)]) 1.1*max([max(tri) max(pulse)])]);
c=conv(pulse,tri)*ts;
subplot(2,1,2)
plot(nn,c);
title('Convolution');
xlabel('Shift (Seconds)');
axis([-1 10 min(c) 1.1*max(c)]);
```

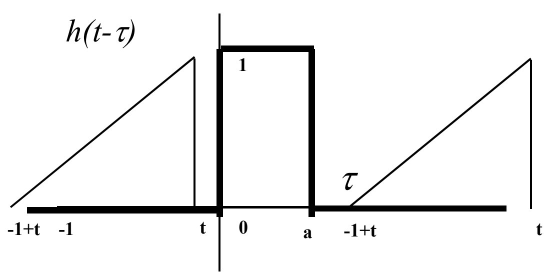


Homework Answers #7

- Convolution
  - Problem (1)
    - Sketch the response to b) *u(t)-u(t-a)* for a=5

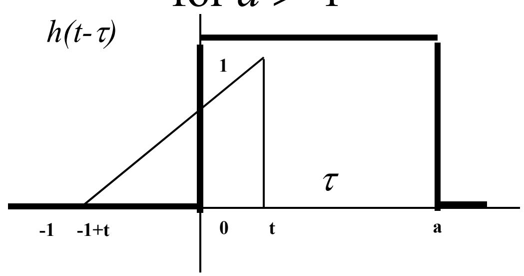
$$b)\ \mathrm{C} = \int [1-(t-\tau)]u(t-\tau)u(\tau)d\tau + \int [1-(t-\tau)]u(t-\tau)u(\tau-a)d\tau$$

for $a > 1$



$t < 0$ & $t > a+1$,

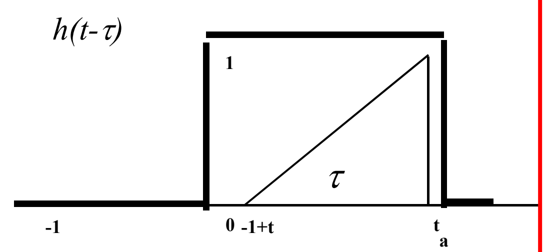
C=0



$0 < t < a < 1$

$$\mathrm{C} = \int_0^t 1*[1-(t-\tau)]d\tau = \left.\frac{[1-(t-\tau)]^2}{2}\right|_0^t$$

$$= \frac{1}{2} - \frac{[1-t]^2}{2} = \frac{1}{2}[1-(1-t)^2]$$



$1 < t < a$

$$\mathrm{C} = \int_{-1+t}^{t} 1*[1-(t-\tau)]d\tau$$

$$= \left.\frac{[1-(t-\tau)]^2}{2}\right|_{-1+t}^{t}$$

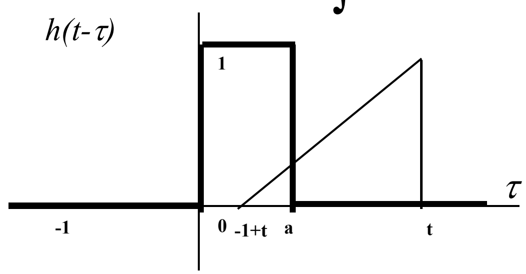
$$= \frac{1}{2} - 0$$

$$= \frac{1}{2}$$

Homework Answers #8

- Convolution
  - Problem (1) :
    - Sketch the response to b) *u(t)-u(t-a)* for a=5

$$b)\ C = \int [1-(t-\tau)]u(t-\tau)u(\tau)d\tau + \int [1-(t-\tau)]u(t-\tau)u(\tau-a)d\tau$$



for $a > 1$

$t < 0;\ C = 0$

$0 < t < a < 1; C = \frac{1}{2}[1-(1-t)^2]$

$1 < t < a; C = \frac{1}{2}$

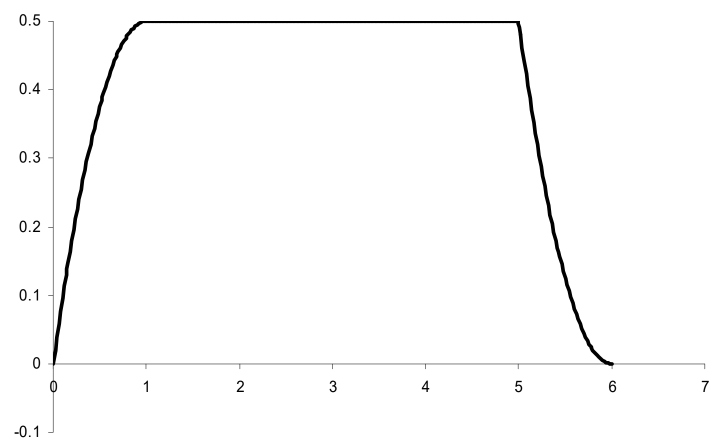
$a < t < a+1$

$$C = \int_{-1+t}^{a} 1*[1-(t-\tau)]d\tau$$

$$= \left.\frac{[1-(t-\tau)]^2}{2}\right|_{-1+t}^{a}$$

$$= \frac{[1-(t-a)]^2}{2} - 0$$

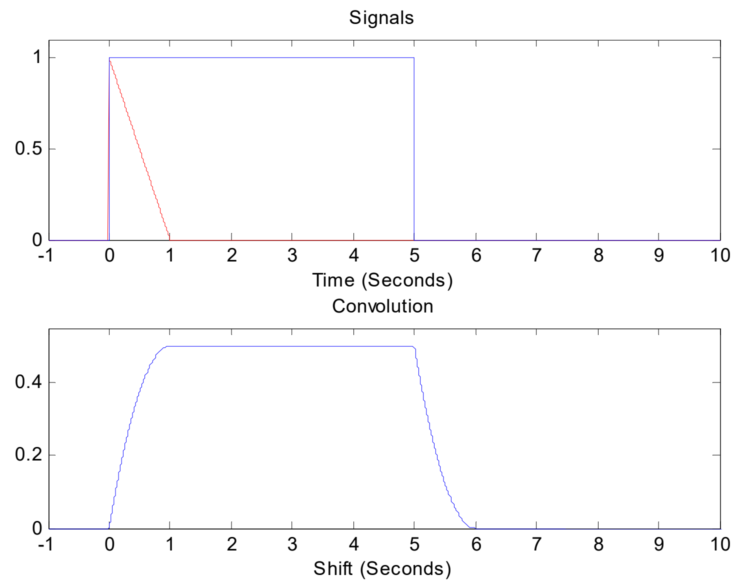
$$= \frac{[1-(t-a)]^2}{2}$$



$a+1;\ C = \frac{[1-(t-a)]^2}{2}$

$C = 0$

Matlab Code

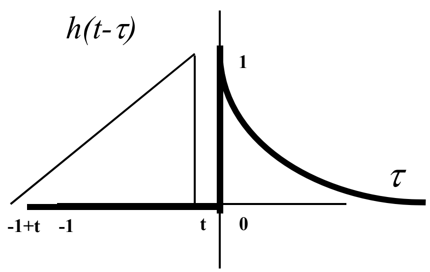


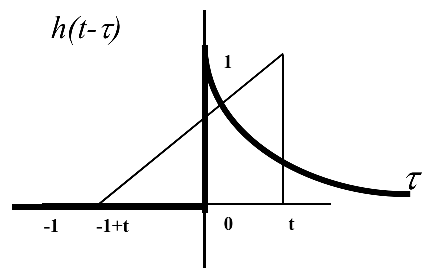

```
clear all;
endpulse=5;
ts=.001;
endpoint=10;
n=-endpoint:ts:endpoint;
nn=-endpoint*2:ts:endpoint*2;
pulser=(n>=0)&(n<=endpulse);
pulse=1*pulser;
tripulse=(n>=0)&(n<=1);
tri=(1-n).*tripulse;
subplot(2,1,1)
plot(n,tri,'r',n,pulse,'b');
title('Signals');
xlabel('Time (Seconds)');
axis([-1 10 min([min(tri) min(pulse)]) 1.1*max([max(tri) max(pulse)])]);
c=conv(pulse,tri)*ts;
subplot(2,1,2)
plot(nn,c);
title('Convolution');
xlabel('Shift (Seconds)');
axis([-1 10 min(c) 1.1*max(c)]);
```

Homework Answers #9

- Convolution
  - Problem (1)
    - c) evaluate $e^{-t}\,u(t)$ at $t=1$ and $t=2$

$$c)\ C=\int[1-(t-\tau)]u(t-\tau)e^{-t}u(\tau)d\tau$$





t<0, C=0

$$0<t<1,\ \ C=\int_0^t e^{-\tau}\times[1-(t-\tau)]d\tau$$

$$=-e^{-\tau}[1-(t-\tau)]-e^{-\tau}\Big|_0^t$$

$$=(-e^{-t}[1-(t-t)]-e^{-t})-(-e^{-0}[1-(t-0)]-e^{-0})$$

$$=(-e^{-t}-e^{-t})-(-1\times[1-t]-1)$$

$$C(t)=2-t-2e^{-t};0<t<1$$

$$C(1)=2-1-2e^{-1}=.264$$

Integration by parts:

$$\int v\frac{du}{d\tau}d\tau=uv-\int u\frac{dv}{d\tau}d\tau$$

$$v=1-(t-\tau);dv=d\tau$$

$$\frac{du}{d\tau}=e^{-\tau};u=-e^{-\tau}$$

$$\int[1-(t-\tau)]e^{-\tau}d\tau$$

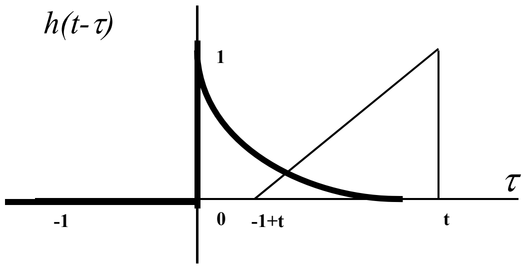
$$=-e^{-\tau}[1-(t-\tau)]+\int e^{-\tau}\times 1d\tau$$

$$=-e^{-\tau}[1-(t-\tau)]-e^{-\tau}+C$$

Homework Answers #10

- Convolution
  - Problem (1)
    - c) evaluate $e^{-t}\,u(t)$ at $t=1$ and $t=2$

$$c)\ C=\int[1-(t-\tau)]u(t-\tau)e^{-t}u(\tau)d\tau$$



$$t>1,\quad C=\int_{-1+t}^{t} e^{-\tau}\times[1-(t-\tau)]d\tau$$

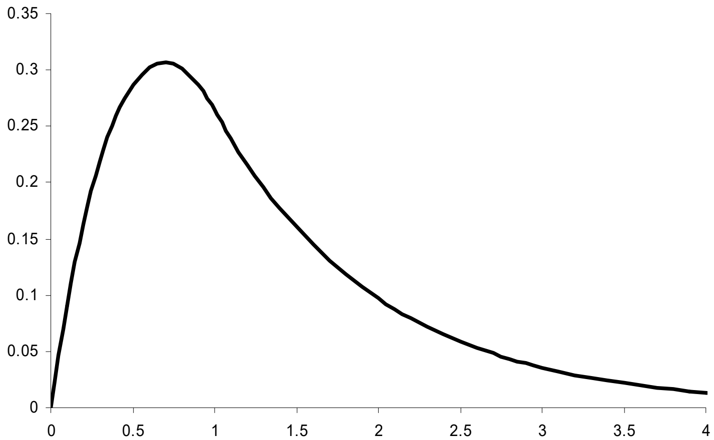
$$=-e^{-\tau}[1-(t-\tau)]-e^{-\tau}\Big|_{-1+t}^{t}$$

$$=(-e^{-t}[1-(t-t)]-e^{-t})-(-e^{1-t}[1-(t-[-1+t])]-e^{1-t})$$

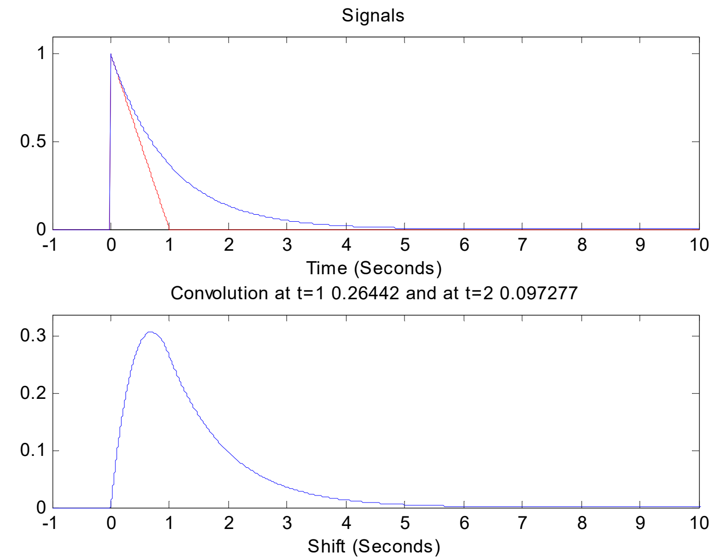
$$=(-e^{-t}[1-(0)]-e^{-t})-(-e^{1-t}[0]-e^{1-t})$$

$$=e^{1-t}-2e^{-t};t>1,$$

$$=e^{-1}-2e^{-2}=.097$$



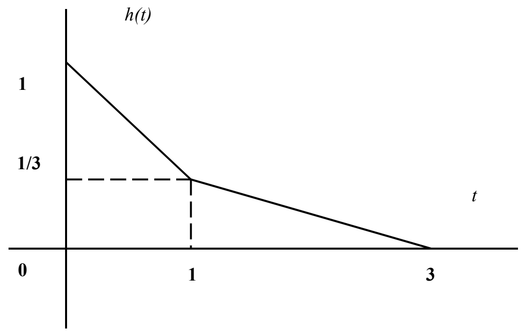
Matlab Code

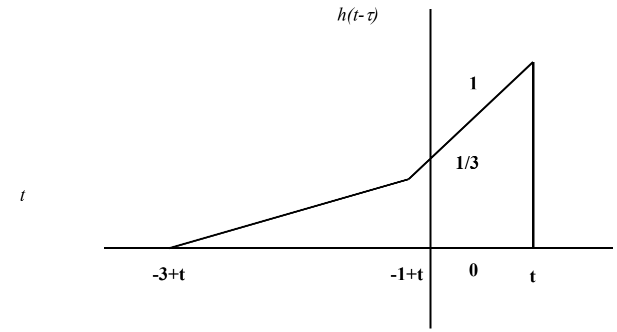



```
clear all;
endpulse=20;
ts=.001;
endpoint=10;
n=-endpoint:ts:endpoint;
nn=-endpoint*2:ts:endpoint*2;
pulser=(n>=0);
pulse=exp(-n).*pulser;
tripulse=(n>=0)&(n<=1);
tri=(1-n).*tripulse;
subplot(2,1,1)
plot(n,tri,'r',n,pulse,'b');
title('Signals');
xlabel('Time (Seconds)');
axis([-1 10 min([min(tri) min(pulse)]) 1.1*max([max(tri) max(pulse)])]);
c=conv(pulse,tri)*ts;
subplot(2,1,2)
plot(nn,c);
for i=1:length(nn)
   if nn(i)==1
      c1=i;
   end
   if nn(i)==2
      c2=i;
   end
end
title(['Convolution at t=1 ',num2str(c(c1)),' and at t=2 ',num2str(c(c2))]);
xlabel('Shift (Seconds)');
axis([-1 10 min(c) 1.1*max(c)]);
```

Homework Answers #11

- Convolution
  - Problem (2)
    - Assume that a system response is given by the following:
    - Evaluate the response to $te^{-t}\,u(t)$ at $t=1$ and $t=3$





$$h(\tau) = (1 - \frac{2}{3}\tau);\ 0 < \tau < 1$$
$$= \frac{1}{6}(3-\tau);\ 1 < \tau < 3$$
$$= 0;\ \text{elsewhere}$$

$$h(t-\tau) = [1 - \frac{2}{3}(t-\tau)];\ 0 < t-\tau < 1, -1+t < \tau < t$$
$$= \frac{1}{6}[3-(t-\tau)];\ 1 < t-\tau < 3, -3+t < \tau < -1+t$$
$$= 0;\ \text{elsewhere}$$

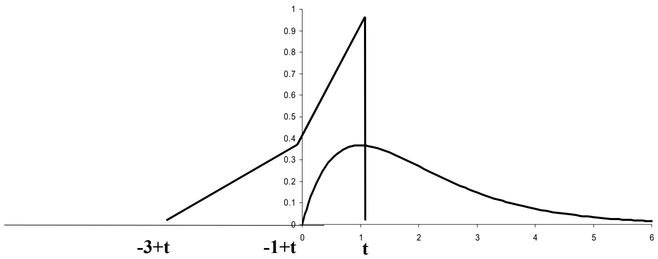
Homework Answers #12

- Convolution
  - Problem (2)
    - Assume that a system response is given by the following:
    - Evaluate the response to $te^{-t}\,u(t)$ at $t=1$ and $t=3$

$$h(t-\tau)=[1-\frac{2}{3}(t-\tau)];\; 0<t-\tau<1, -1+t<\tau<t$$

$$=\frac{1}{6}[3-(t-\tau)];\; 1<t-\tau<3, -3+t<\tau<-1+t$$

$$=0;\; \text{elsewhere}$$



$$C=\int_0^t \tau e^{-\tau}h(t-\tau)d\tau$$

$$=\int_0^t \tau[1-\frac{2}{3}(t-\tau)]e^{-\tau}d\tau$$

$$=\int_0^1 \tau[1-\frac{2}{3}(1-\tau)]e^{-\tau}d\tau$$

$$=\int_0^1 \tau[\frac{1}{3}+\frac{2}{3}\tau]e^{-\tau}d\tau$$

$$C=\int_0^1 [\frac{2}{3}\tau^2+\frac{1}{3}\tau]e^{-\tau}d\tau$$

Homework Answers #13

$$C = \int_0^1 [\frac{2}{3}\tau^2 + \frac{1}{3}\tau]e^{-\tau}d\tau$$

$$= \{\frac{2}{3}(\tau^2 + 2\tau + 2) + \frac{1}{3}(\tau + 1)\}(-e^{-\tau})\Big|_0^1$$

$$= \{\frac{2}{3}(1+2+2) + \frac{1}{3}(1+1)\}(-e^{-1})$$

$$-\{\frac{2}{3}(0+0+2) + \frac{1}{3}(0+1)\}(-e^{-0})$$

$$= \frac{5}{3} - [\frac{12}{3}(-e^{-1})]$$

$$= .195$$

Integration by parts

$$\int_a^b t^2 e^{-t} dt$$

$$u = t^2; du = 2tdt$$

$$dv = e^{-t}dt; v = -e^{-t}$$

$$\int_a^b t^2 e^{-t} dt = t^2(-e^{-t})\Big|_a^b - (-2)\int_a^b te^{-t})dt$$

$$= t^2(-e^{-t})\Big|_a^b - (-2)[(t+1)(-e^{-t})\Big|_a^b]$$

$$= (t^2 + 2t + 2)(-e^{-t})\Big|_a^b$$

$$\int_a^b te^{-t} dt$$

$$u = t; du = 1dt$$

$$dv = e^{-t}dt; v = -e^{-t}$$

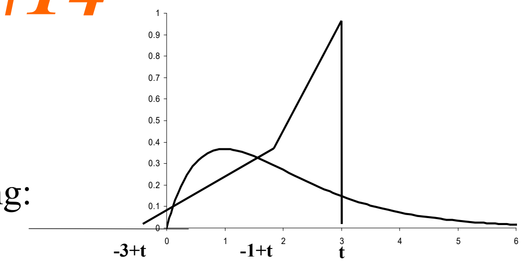
$$\int_a^b te^{-t} dt = t(-e^{-t})\Big|_a^b - \int_a^b (1)(-e^{-t})dt$$

$$= t(-e^{-t})\Big|_a^b - e^{-t}\Big|_a^b$$

$$= (t+1)(-e^{-t})\Big|_a^b$$

Homework Answers #14

- Convolution
  - Problem (2)
    - Assume that a system response is given by the following:
    - Evaluate the response to $te^{-t}\,u(t)$ at $t=1$ and $t=3$



$$C=\int_0^t \tau e^{-\tau}h(t-\tau)d\tau$$

$$=\int_{-1+t}^{t}\tau[1-\frac{2}{3}(t-\tau)]e^{-\tau}d\tau$$

$$+\int_0^{-1+t}\frac{1}{6}\tau[3-(t-\tau)]e^{-\tau}d\tau$$

$$=\int_2^3\tau[1-\frac{2}{3}(3-\tau)]e^{-\tau}d\tau$$

$$+\int_0^2\frac{1}{6}\tau[3-(3-\tau)]e^{-\tau}d\tau$$

$$=\int_2^3[\frac{2}{3}\tau^2-\tau]e^{-\tau}d\tau+\int_0^2\frac{\tau^2}{6}e^{-\tau}d\tau$$

$$C=\int_2^3[\frac{2}{3}\tau^2-\tau]e^{-\tau}d\tau+\int_0^2\frac{\tau^2}{6}e^{-\tau}d\tau$$

$$=\frac{2}{3}(\tau^2+2\tau+2)(-e^{-\tau})\Big|_2^3-(\tau+1)(-e^{-\tau})\Big|_2^3+\frac{1}{6}(\tau^2+2\tau+2)(-e^{-\tau})\Big|_0^2$$

$$=\{\frac{2}{3}(9+6+2)(-e^{-3})-(3+1)(-e^{-3})+\frac{1}{6}(4+4+2)(-e^{-2})\}$$

$$-\{\frac{2}{3}(4+4+2)(-e^{-2})-(2+1)(-e^{-2})+\frac{1}{6}(0+0+2)(-e^{-0})\}$$

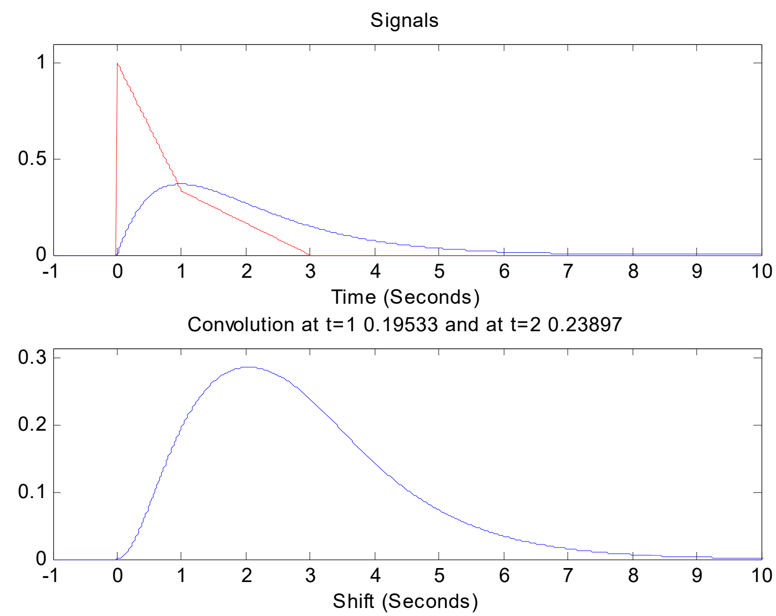
$$=\{\frac{22}{3}(-e^{-3})+\frac{5}{3}(-e^{-2})\}-\{\frac{11}{3}(-e^{-2})+\frac{1}{3}(-e^{-0})\}$$

$$=\frac{1}{3}\{1+6e^{-2}-22e^{-3}\}$$

$$=.239$$

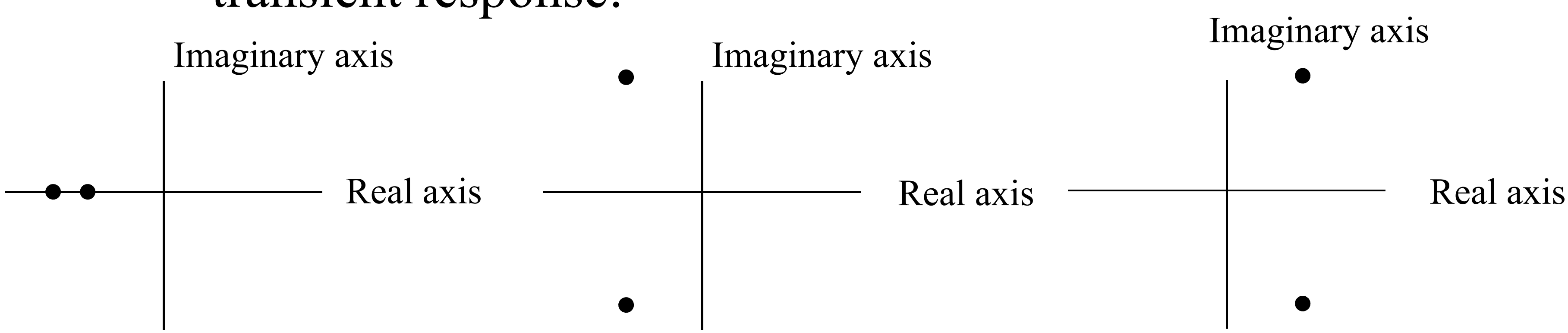
Matlab Code

```
clear all;
endpulse=20;
ts=.001;
endpoint=10;
n=-endpoint:ts:endpoint;
nn=-endpoint*2:ts:endpoint*2;
pulser=(n>=0);
pulsers=exp(-n).*pulser;
pulse=n.*pulsers;
tripulse=(n>=0)&(n<=1);
tri1=(1-2*n/3).*tripulse;
tripulse2=(n>1)&(n<=3);
tri2=1/6*(3-n).*tripulse2;
tri=tri1+tri2;
subplot(2,1,1)
plot(n,tri,'r',n,pulse,'b');
title('Signals');
xlabel('Time (Seconds)');
axis([-1 10 min([min(tri) min(pulse)]) 1.1*max([max(tri) max(pulse)])]);
c=conv(pulse,tri)*ts;
subplot(2,1,2)
plot(nn,c);
for i=1:length(nn)
   if nn(i)==1
      c1=i;
   end
   if nn(i)==3
      c2=i;
   end
end
title(['Convolution at t=1 ',num2str(c(c1)),' and at t=2 ',num2str(c(c2))]);
xlabel('Shift (Seconds)');
axis([-1 10 min(c) 1.1*max(c)]);
```



Homework Answers #15

- Stability
  - Determine the stability of the following systems with poles in the complex plane, describe the form of the transient response:



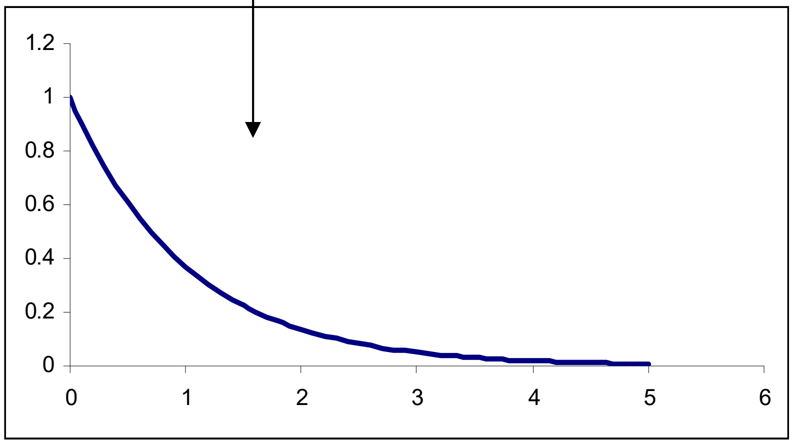
Stable Overdamped system	Stable Underdamped system	Unstable system Oscillatory with Exponentially increasing Amplitude

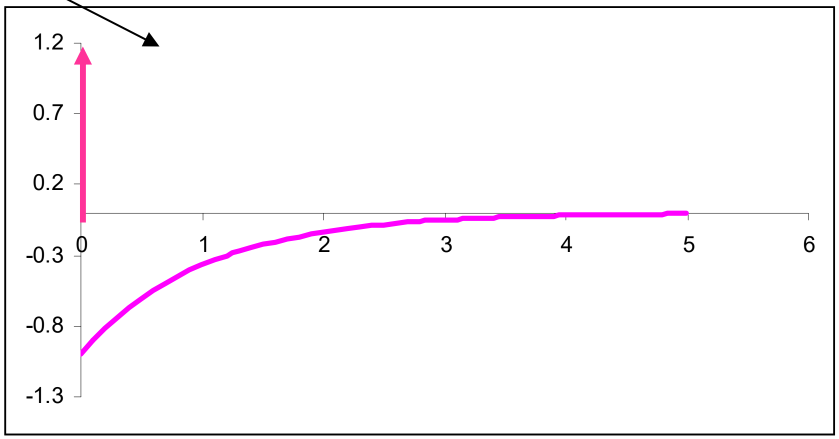
2CT3.1

$$x(t) = e^{-t/\tau}u(t)$$

$$y(t) = \frac{dx(t)}{dt} = \frac{d\{e^{-t/\tau}u(t)\}}{dt} = u(t)\frac{d\{e^{-t/\tau}\}}{dt} + e^{-t/\tau}\frac{d\{u(t)\}}{dt}$$

$$= -\frac{u(t)e^{-t/\tau}}{\tau} + e^{-t/\tau}\delta(t) = e^{-0/\tau}\delta(t) - \frac{1}{\tau}e^{-t/\tau}u(t) = \delta(t) - \frac{1}{\tau}e^{-t/\tau}u(t)$$

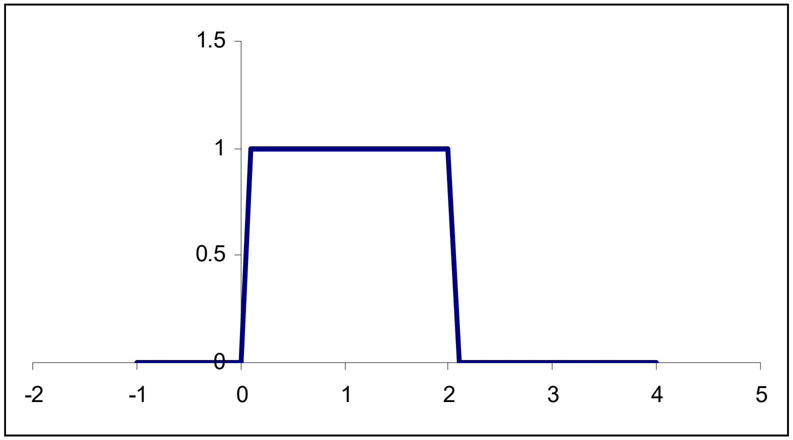




2CT3.2

$$y(t) = \frac{1}{\tau}\int_{t-\tau}^{t} x(\lambda)d\lambda$$

$$h(t) = \frac{1}{\tau}\int_{t-\tau}^{t} \delta(\lambda)d\lambda;$$



$Note:$

$$u(t) = \int_{-\infty}^{t} \delta(\lambda)d\lambda$$

$$h(t) = \frac{1}{\tau}\int_{t-\tau}^{t} \delta(\lambda)d\lambda = \frac{1}{\tau}\int_{-\infty}^{t} \delta(\lambda)d\lambda - \frac{1}{\tau}\int_{-\infty}^{t-\tau} \delta(\lambda)d\lambda$$

$$= \frac{1}{\tau}u(t) - \frac{1}{\tau}u(t-\tau)$$